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SOME SINGULAR MODULI FOR $\mathbf{Q}(\sqrt{3})$

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ABSTRACT. In an earlier paper in this journal, the authors derived the equations which transform the Hilbert modular function field for $\mathbf{Q}(\sqrt{3})$ when the arguments are multiplied by $(1+\sqrt{3}, 1-\sqrt{3})$. These equations define a complex V_2, but we concentrate on special diagonal curves on which the values of some of the singular moduli can be evaluated numerically by using the "PSOS" algorithm. In this way the ring class fields can be evaluated for the forms $\xi^2 + 2^t A\eta^2$, where $A = 1, 2, 3, 6$ and $t > 0$. These last results are based partly on conjectures supported here by numerical evidence.

1. INTRODUCTION

The historical motivation of modular functions in several variables was to generalize some powerful results in one variable to the effect that singular moduli determine class fields. This thesis of Hilbert led to Hecke [9], and it was not easily realized for numerical results because of the escalated difficulties of computation. These difficulties are partly due to the fact that Fourier series are now doubly infinite, but mostly due to the paucity of modular equations (see [4, 6]). The evaluation of numerical constants is more precarious, but the "PSOS" routines [1] are remarkably effective in identifying algebraic numbers.

The earlier papers ([4, 5]) treated some special cases for modular functions over the field $\mathbf{Q}(\sqrt{2})$. They led to theorems on the representation of primes in terms of integral quadratic forms in this field,

$$\pi = \xi^2 + 2^t\eta^2, \quad \text{or} \quad \pi = \xi^2 + (2+\sqrt{2})2^t\eta^2. \tag{1.1}$$

By using modular equations in [6], we now do the same for $\mathbf{Q}(\sqrt{3})$, e.g., with the forms

$$\pi = \xi^2 + A2^t\eta^2, \qquad A = 1,\ 2,\ 3,\ 6. \tag{1.2}$$

The definitions will be reviewed at the appropriate point in this paper, but to summarize, the theory of Weber, which worked to perfection for the case of the Klein modular group, is seriously limited when Hecke's thesis is applied. In fact, we restrict ourselves to forms with a normal splitting field (see §8), thus avoiding a good deal of the complexity. Indeed, the theory is still conjectural,

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but it is hoped that this paper will add to the numerical evidence showing an analogue.

2. Review of equations and notations

The principal equations of [6] are reproduced here for convenience. We are dealing with the Hilbert modular group $PGL_2^+(\mathbf{Z}[\sqrt{3}])$ as it acts simultaneously on two products of half-planes H^+ and H^- as follows:

$$Z = (z, z'), \quad \Im z > 0 \quad \text{and} \quad Z \in \begin{cases} H^+ & \text{when } \Im z' > 0 \\ H^- & \text{when } \Im z' < 0. \end{cases} \tag{2.1}$$

The algebraic numbers $\beta \in \mathbf{Q}(\sqrt{3})$ are defined, with conjugates, by

$$\beta = a + b\sqrt{3}, \qquad \beta' = a - b\sqrt{3} \qquad (a, b \in \mathbf{Q}), \tag{2.2a}$$

$$Z + \beta = (z + \beta, z' + \beta'), \qquad \beta Z = (\beta z, \beta' z'). \tag{2.2b}$$

The latter assumes that $\beta > 0$. Furthermore, if $\beta' > 0$, then the relation $Z \in H^{\pm}$ is preserved by βZ, otherwise that relation is reversed. In particular, note

$$\varepsilon = 2 + \sqrt{3}, \qquad \tau = 1 + \sqrt{3}, \tag{2.2c}$$

where $\varepsilon' > 0$, $\tau' < 0$. Indeed, $N\varepsilon = 1$ and $N\tau = -2$. Also $\tau^2 = 2\varepsilon$, or equivalently, $\sqrt{\varepsilon} = \tau/\sqrt{2}$. We also write $1/Z = (1/z, 1/z')$. The modular group is then generated (see [8]) by

$$Z = Z + 1, \quad Z = Z + \sqrt{3}, \quad Z = -1/Z, \quad Z = \varepsilon Z. \tag{2.2d}$$

For the products of half-planes, we further impose the symmetry (which augments the group $PGL_2^+(\mathbf{Z}[\sqrt{3}])$ by index two)

$$H^+ \colon z \to z', \quad z' \to z, \tag{2.3a}$$

$$H^- \colon z \to -z', \quad z' \to -z. \tag{2.3b}$$

We have previously defined (see [6]) field-generating modular functions (chosen for their "simple poles"):

$$U = U(Z), \qquad V = V(Z) \quad \text{for } Z \in H^+, \tag{2.4a}$$

$$X = X(Z), \qquad Y = Y(Z) \quad \text{for } Z \in H^-. \tag{2.4b}$$

We also need

$$W(Z) = U(Z)V(Z), \qquad Z \in H^+, \tag{2.4c}$$

where, in terms of modular forms of indicated dimension,

$$U = {H_2^+}^2/H_4^+, \quad V = H_2^+ H_4^+/H_6^+, \quad W = {H_2^+}^3/H_6^+, \tag{2.4d}$$

$$X = {H_2^-}^2/H_4^-, \quad Y = H_2^- H_4^-/H_6^-. \tag{2.4e}$$

Now we move to the transformation equations themselves (see [6]):

$$\begin{aligned} &(-X^3 - 144X^2 - 5184X)Y^2 + (-3456X^2 - 248832X)Y - 2985984X \\ &+ ((2X^2 + 288X + 10368)Y^2 + (-207X^2 - 9504X)Y + 62208X)U \\ &+ ((-X^2 + 78X)Y - 432X)U^2 + XU^3 = 0, \end{aligned} \tag{2.5a}$$

$$\begin{aligned}(-U^3+864U^2-186624U)V^2+(3456U^2-1492992U)V-2985984U\\ +((-87U^2+2592U)V^2+(-414U^2-20736U)V-124416U)X\\ +((U^2-72U)V^2+(4U^2-432U)V-1728U)X^2\\ +((-U+32)V^2-6UV-8U)X^3=0,\end{aligned} \tag{2.5b}$$

$$\begin{aligned}(16X^2+1152X)Y+27648X+(-20XY^2+(20X^2+96X)Y+13824X)V\\ +(4Y^3+(-12X+96)Y^2+(8X^2-160X)Y+2304X)V^2\\ +(Y^3+(-2X+24)Y^2+(X^2-24X)Y+128X)V^3=0,\end{aligned} \tag{2.5c}$$

$$\begin{aligned}(U^2-464U+13824)V^3+(6U^2-4320U+55296)V^2\\ +(8U^2-13824U)V-13824U\\ +((71U+1728)V^3+(360U+6912)V^2+432UV)Y\\ +((-U+72)V^3+(-2U+288)V^2)Y^2+(V^3+4V^2)Y^3=0.\end{aligned} \tag{2.5d}$$

Thus, given $U(Z)$ and $V(Z)$, the equations (2.5a–d) define $X(Z^*)$ and $Y(Z^*)$, and given $X(Z)$ and $Y(Z)$, they define $U(Z^*)$ and $V(Z^*)$, where Z^* takes the three values

$$Z^*=Z\tau,\quad Z/\tau,\quad (Z+1)/\tau. \tag{2.6}$$

(This is strongly analogous to the classical case where the modular equation of order two for $j(z)$ defines $j(2z)$, $j(z/2)$, and $j((z+1)/2)$.) Note that, if $Z\in H^+$, then $Z^*\in H^-$, and conversely. The four equations in U, V, X, Y form a two-dimensional variety with diagonal curves lying on it (as we see next).

3. Some diagonal identities for modular forms

We next consider the modular functions as confined to diagonals of the form

$$z/z'=\pm\varepsilon^t,\qquad t\in\mathbf{Z}. \tag{3.1a}$$

Here, of course (by equations (2.2d)), there are only four cases to consider under modular equivalence. They arise from the $\pm$ and the choice of t odd or even. The set of Z satisfying (3.1a) are invariant under the transformation

$$T[Z]=\tau Z \tag{3.1b}$$

(recall that $\varepsilon=\tau/\tau'$). Therefore, the diagonals can be partitioned into two subsets equivalent to

$$D:\ (z,-z),\quad (z\tau,-z\tau') \tag{3.1c}$$

or to

$$D^*:\ (z,z),\quad (z\tau,z\tau'). \tag{3.1d}$$

Each is preserved under modular equivalence by the transformation T, and the iteration of T amounts to $z\to 2z$, since $\tau^2=2\varepsilon$. Each diagonal set will be seen to correspond to a curve on the projective manifold of the modular equations (§5).

These diagonal classes lead to the new functions on the upper-half z-plane,

(3.2a) $$D: \ x(z) = X(z, -z), \ y(z) = Y(z, -z),$$

(3.2b) $$D: \ u(z) = U(z\tau, -z\tau'), \ v(z) = V(z\tau, -z\tau').$$

Here, we have the Hecke modular relation for $f(z) = x(z)$ or $y(z)$:

(3.2c) $$f(z + \sqrt{3}) = f(z), \qquad f(-1/z) = f(z).$$

There is also a second set requiring the use of W $(= UV)$,

(3.3a) $$D^*: \ u^*(z) = U(z, z), \ v^*(z) = V(z, z), \ w^*(z) = W(z, z),$$

(3.3b) $$D^*: \ x^*(z) = X(z\tau, z\tau'), \ y^*(z) = Y(z\tau, z\tau').$$

It is now seen that *formally* $f(z) = u^*(v)$, $v^*(z)$, or $w^*(z)$ satisfies the Klein modular relation

(3.3c) $$f(z + 1) = f(z), \qquad f(-1/z) = f(z).$$

(We shall see that only $w^*(z)$ is finite, by (3.7c) below.)

By this same procedure, we define modular forms on the diagonals, valid on the upper-half z-plane. Analogously with (3.2a–c),

(3.4a) $$k_m(z) = H_m^-(z, -z), \qquad k_m^*(z) = H_m^+(z\tau, -z\tau'),$$

(3.4b) $$k_m(z + \sqrt{3}) = k_m(z), \qquad k_m(-1/z) = k_m(z)z^{2m}.$$

Likewise, analogously with (3.3a–c),

(3.5a) $$h_m(z) = H_m^+(z, z), \qquad h_m^*(z) = H_m^-(z\tau, z\tau'),$$

(3.5b) $$h_m(z + 1) = h_m(z), \qquad h_m(-1/z) = h_m(z)/z^{2m}.$$

This diagonal reduction is applied to the $H_m^{\pm}$ of (2.4d,e) and as well to two additional functions required here,

(3.6) $$H_8^- = H_4^{-2} - H_2^- H_6^-, \qquad H_{12}^+ = H_6^{+2} - 4H_4^{+3}.$$

To reduce to one complex dimension, we eliminate one degree of freedom in the generators (2.4a,b) of the modular function fields. This is a result of the following identities in z over D:

(3.7a) $$D: \ x(z) = y(z), \ u(z) = 4v^2(z),$$

or, equivalently,

(3.7b) $$D: \ k_8(z) = 0, \ k_{12}^*(z) = 0.$$

Likewise, over D^*:

(3.7c) $$D^*: \ u^*(z) = \infty, \ y^*(z) = \infty,$$

or, equivalently,

(3.7d) $$D^*: \ h_4(z) = 0, \ h_6^*(z) = 0.$$

These conjectures, particularly (3.7a), resulted from numerical computations of §6 (below).

4. Use of Fourier series

Each of the identities in (3.7b) and (3.7d) can be proved by verifying just that a finite number of the leading terms of each Fourier series vanish.

In the earlier work [6] (also see [13]), the various modular forms $H_t^{\pm}$ were expanded into Fourier series (over $\mathbf{Z}$) in

(4.1a) $$Q = \exp \pi i(z + z'), \qquad R = \exp \pi i(z - z')/\sqrt{3},$$

so we can rewrite

(4.1b) $$H_m^{\pm}(z, z') = H_m^{\pm}[Q, R] = \sum C_{a,b} Q^b R^a$$

with indices delimited by

(4.1c) $$|a| \leq b\sqrt{3} \quad \text{for } H_m^+,$$

(4.1d) $$|b| \leq a/\sqrt{3} \quad \text{for } H_m^-.$$

The coefficients $C_{a,b}$ appear in tables in [6]. We define the *degree of accuracy* of a section of the sum (4.1b) as the maximum value of b for H_m^+ and of a for H_m^-.

For purposes of diagonalization, we use the new variables (observe difference in lower-case lettering compared with [6]):

(4.2a) $$q = \exp 2\pi i z, \qquad r = \exp 2\pi i z/\sqrt{3}.$$

Here, q is the parameter at ∞ for $z \to z + 1$, and r is for $z \to z + \sqrt{3}$. We verify that, referring to (3.4a) and (3.5a),

(4.2b) $$D: \ k_m(z) = H_m^-[1, r], \quad k_m^*(z) = H_m^+[r^3, r],$$

(4.2c) $$D^*: \ h_m(z) = H_m^+[q, 1], \quad h_m^*(z) = H_m^+[q, q].$$

Our main purpose for now is to verify the identities in (3.7b,d). Actually, $h_4(z) = 0$ and $h_6^*(z) = 0$ are classic (see [8]), and the same methods may be applied to show the identities $k_8(z) = 0$ and $k_{12}^*(z) = 0$. To summarize this technique, note that $k_8(z)$ is a cusp form of weight 16 for the Hecke modular group (see (3.4b)). It can also be verified that

(4.3) $$K_{12}(z) = k_{12}^*(z) k_{12}^*(z/2) k_{12}^*((z+1)/2)$$

is a cusp form of weight 72. By classical theorems (see [8]) it follows that if a cusp form of weight $2m$ for the Hecke modular group vanishes at ∞ of order r^n for $n > m/3$, then it vanishes identically. (We note that $K_{12}(z)$ has *twice* the order of $k_{12}^*(z)$. Of course, if K_{12} vanishes identically, so does *each* of the factors as images under the Hecke modular group.)

Therefore, to complete the proofs of identities (3.7b,d), we have only to show that each of the following Fourier series,

(4.4a) $$k_8(z) = H_4^-[1, r]^2 - H_2^-[1, r] H_6^-[1, r]$$

and

(4.4b) $$k_{12}^*(z) = H_6^+[r^3, r]^2 - 4H_4^+[r^3, r]^3,$$

vanishes to higher order than r^6. This requirement is easily satisfied if the functions H_m^- are calculated to a degree of accuracy exceeding 6 in (4.4a) and exceeding $6/(3-\sqrt{3})$ in (4.4b) (compare (4.1c,d)). The tables in [6] have somewhat higher accuracy, and indeed accuracy of degree exceeding 35 was regularly used to obtain adequate decimal precision for irrationals.

5. The diagonal curves

The diagonal manifolds D and D^* define diagonal curves which shall be designated the same way.

CURVE D

From (3.7a) we substitute

$$X = x, \quad Y = x, \quad V = v, \quad U = 4v^2 \tag{5.1a}$$

into the system (2.5a–d). We find that the greatest common divisor of all four equations is the first diagonal curve D:

$$F(v, x) = x^2 - 4v^2x + 6vx + 72x + 8v^3 + 144v^2 + 864v + 1728. \tag{5.1b}$$

This is a curve of genus zero. For example, it is quadratic in x with a discriminant having multiple roots,

$$\Delta = (2v+3)^2(v-12)(v+4). \tag{5.1c}$$

The curve may therefore be parametrized by making $(v-12)(v+4)$ a perfect square:

$$v = \phi(t) = \frac{12+4t^2}{1-t^2}, \tag{5.1d}$$

$$x = \psi(t) = \frac{8t^3 + 72t^2 + 216t + 216}{t^3 - t^2 - t + 1}. \tag{5.1e}$$

We want a further parametrization of t to take care of the fact that for each x there are three values v, v', v''. Therefore, if $F(v, x) = 0$ for a given (v, x), then for the same x the roots of a quadratic in (say) V will determine v' and v'':

$$8V^2 + V(8v + 144 - 4x) + (8v^2 + 144v - 4vx + 864 + 6x) = 0. \tag{5.2a}$$

In terms of the parametrization by t in (5.1d,e), the discriminant of the above equation is

$$\Delta' = \frac{[(t+3)(7t+9)]^2}{(t-1)^4}\,\frac{3-t}{1+t}. \tag{5.2b}$$

We introduce new parameters to make $(3-t)/(1+t)$ a perfect square:

$$t = \sigma(s) = \frac{3-s^2}{1+s^2}. \tag{5.2c}$$

So, corresponding to one of two roots $\pm s$ in (5.2c), v' is parametrized by (5.1d) and (5.2c) as

$$v' = \frac{s^4 + 18s^2 + 24s + 21}{s^3 - s^2 - s + 1} = \phi(t'). \tag{5.2d}$$

This yields two values of t' of which the nonextraneous one is

$$t' = -\frac{s^2 - 6s - 3}{s^2 + 2s + 5} = \sigma(s'), \tag{5.2e}$$

where s' is found from (5.2c) to be

$$s' = \omega(s) = \sqrt{\frac{s^2 + 3}{2(s + 1)}}. \tag{5.2f}$$

With proper choice of conjugates, v represents $V(z\tau, z\tau')$, and v' represents $V(z/\tau, z/\tau')$. Thus, $\{v' \to v\}$ amounts to multiplication of Z by τ^2 and of z by 2. (It is more convenient for notation to interchange the conjugates v and v'.)

At last we achieve a parametrization in s for D with the transition (see (3.2a,b))

$$\{v \to v'\} \equiv \{z \to 2z\} \equiv \{s \to \omega(s)\}. \tag{5.2g}$$

CURVE D^*

From (3.7c) we substitute

$$U = w^*/v^*, \quad V = v^*, \quad X = x^*, \quad Y = 1/L \tag{5.3a}$$

into the set of equations (2.5a–d) and rationalize denominators. We then set $v^* = 0$ and $L = 0$ (treating the equations projectively). The equation (2.5b) yields D^*:

$$\begin{aligned} F^*(x^*, w^*) = w^{*2} - 4x^{*2}w^* + 414x^*w^* - 3456w^* + 8x^{*3} \\ + 1728x^{*2} + 124416x^* + 2985984. \end{aligned} \tag{5.3b}$$

The other modular equations are trivial under the substitution and limit in (5.3a). Equation (5.3b) is also of genus zero. It is a quadratic in w^* with discriminant

$$\Delta^* = 4x^*(x^* - 128)(2x^* - 81)^2. \tag{5.3c}$$

As before, we parametrize $x^*(x^* - 128)$ into a perfect square by

$$x^* = \phi^*(t^*) = \frac{128}{1 - t^{*2}}, \tag{5.3d}$$

$$w^* = \psi^*(t^*) = \frac{1728t^{*3} + 8640t^{*2} + 14400t^* + 8000}{t^{*3} - t^{*2} - t^* + 1}. \tag{5.3e}$$

For any root (x^*, w^*), there are two other roots $x^{*\prime}$ and $x^{*\prime\prime}$ which satisfy a quadratic in (say) X,

$$\begin{aligned} 8X^2 + X(1728 - 4w^* + 8x^*) \\ + (124416 + 414w^* - 4w^*x^* + 1728x^* + 8x^{*2}) = 0. \end{aligned} \tag{5.4a}$$

In terms of the parametrization, the determinant becomes

$$\Delta^{*\prime} = \frac{32(3t^* + 5)^2(63t^* + 65)^2}{(t^* - 1)^4(t^* + 1)}. \tag{5.4b}$$

The new parameter for t^* is

$$t^* = \sigma^*(s^*) = 2s^{*2} - 1. \tag{5.4c}$$

So, corresponding to one of the two roots $\pm s^*$ in (5.4c), $x^{*\prime}$ is parametrized by (5.3d) and (5.4c) as

$$x^{*\prime} = \frac{4(3s^* + 1)^4}{(s^* - 1)^2 s^*(s^* + 1)} = \phi^*(t^{*\prime}). \tag{5.4d}$$

This yields two values of $t^{*\prime}$ of which the nonextraneous one is

$$t^{*\prime} = +\frac{7s^{*2} + 10s^* - 1}{(3s^* + 1)^2} = \sigma^*(s^{*\prime}), \tag{5.4e}$$

where $s^{*\prime}$ is found from (5.4c) to be

$$s^{*\prime} = \omega^*(s^*) = \frac{\sqrt{8s^*(s^* + 1)}}{3s^* + 1}. \tag{5.4f}$$

Likewise, in analogy with (5.2g), with a suitable relabeling of conjugates, we achieve a parametrization in s^* for D^* with the transitions (see (3.3ab))

$$\{x^* \to x^{*\prime}\} \equiv \{z \to 2z\} \equiv \{s^* \to \omega^*(s^*)\}. \tag{5.4g}$$

Incidentally, it is a classic result [8] that $w^*(z)$ of (3.3a) is $j(z)$, so the modular equation can be derived by eliminating x^* from the pair:

$$F^*(x^*, j(z)) = F^*(x^*, j(2z)) = 0. \tag{5.5}$$

6. Specific singular moduli

We are concerned with computing specific moduli which are useful to ring class field theory (see §8). The term *singular moduli* is defined by extension of the concepts for classical modular functions as a value of a function in the modular function field for a quadratic surd argument over the real base field, provided the surd is (totally) imaginary. These are the following sequences for $A = 1, 2, 3, 6$ (and $n \in \mathbf{Z}^+$):

The data from Curve D are abbreviated as

$$\begin{aligned} D:\ v_n[A] &= v(2^{n-1}\sqrt{-A}) = V(2^{n-1}\tau\sqrt{-A}, -2^{n-1}\tau'\sqrt{-A}) \\ &= V(\tau^{2n-1}\sqrt{-A}, -\tau'^{2n-1}\sqrt{-A}), \end{aligned} \tag{6.1a}$$

$$\begin{aligned} D:\ x_n[A] &= x(2^n\sqrt{-A}) = X(2^n\sqrt{-A}, -2^n\sqrt{-A}) \\ &= X(\tau^{2n}\sqrt{-A}, -\tau'^{2n}\sqrt{-A}). \end{aligned} \tag{6.1b}$$

Omitting the "$[A]$" when the value is understood, we use the parametrizations of §5, namely,

$$v_n = \phi(t_n), \qquad x_n = \psi(t_n), \tag{6.1c}$$

$$t_n = \sigma(s_n), \qquad s_{n+1} = \omega(s_n). \tag{6.1d}$$

The data for Curve D^* are likewise abbreviated as

$$\begin{aligned} D^*:\ x_n^*[A] &= x^*(2^{n-1}\sqrt{-A}) = X(2^{n-1}\tau\sqrt{-A}, 2^{n-1}\tau'\sqrt{-A}) \\ &= X(\tau^{2n-1}\sqrt{-A}, \tau'^{2n-1}\sqrt{-A}), \end{aligned} \tag{6.2a}$$

$$(6.2b) \qquad D^*: \; w_n^*[A] = w^*(2^n\sqrt{-A}) = W(2^n\sqrt{-A}, 2^n\sqrt{-A}) = W(\tau^{2n}\sqrt{-A}, \tau'^{2n}\sqrt{-A}).$$

The parametrizations likewise are

$$(6.2c) \qquad x_n^* = \phi^*(t_n^*), \qquad w_n^* = \psi^*(t_n^*),$$

$$(6.2d) \qquad t_n^* = \sigma^*(s_n^*), \qquad s_{n+1}^* = \omega^*(s_n^*).$$

In either case it is clear that s_0 or s_0^* determines the whole sequence. The problem is to *start* the process. The values of the sequences themselves, (6.1a,b) and (6.2a,b), are of course well defined, as well as the uniformizing parameters t or t^*. There could be an ambiguity in the choice of s and s^*, which extends itself to the sequences s_n and s_n^*. We note, however, that the choice of (say) v' over v'' in (5.2a) may be taken as the *numerically larger* value. This can be systematically done by taking s_n and s_n^* to be positive. Yet the choice is of no importance *algebraically,* since we are dealing with *normal* fields (see §8). Each sequence will approach 1 monotonically from below or above. There were enough "recognizable" integers to guess some of the moduli. It was still desirable, however, to evaluate some of the surds in exact form from the decimal expansion from the PSOS algorithm [1] for verification. (This is sketched in §7.)

We list the sequences of points (v_n, x_n) on D and (x_n^*, w_n^*) on D^* (as defined by (6.1ab) and (6.2ab)) at the top of each of Tables I–IV (for $A = 1, 2, 3, 6$, respectively). These values came numerically from power series in [6] and were verified by the PSOS algorithm.

In the context of singular moduli belonging to fixed points of quadratic transformations, we note that $v = 12$ and $x^* = 128$ are roots of the discriminants Δ and Δ^* in §5. This is not surprising, since, e.g., if $Z = (i, \pm i)$, two of the three conjugates in (2.6) are equivalent (namely Z/τ and $Z\tau$) under the modular group.

7. Computational considerations

The evaluation of singular moduli was aided by symbolic computations in a number of ways. One direct method was to approximate singular moduli using expansions of modular functions, and then solve (2.5a–d) iteratively. If (say) x and y were known, then one of these four equations would have $x, y,$ and u as variables, and another would have $x, y,$ and v as variables. Clearing radicals if necessary, we could solve the resulting equations for u and v by factoring over appropriate number fields. Such factoring was accomplished by using an option of the MACSYMA factor command. Radical expressions were thus generated for the singular moduli, which could be matched with decimal approximations with some degree of confidence. The entire process could be repeated, starting with the values obtained for u and v. Since radical expressions progressively increased in complexity, the entire procedure quickly became too involved.

A few of the singular moduli were obtained by use of the LLL algorithm. The algorithm was written in MACSYMA, and runs of LLL in MACSYMA generally took less than seven minutes. In this algorithm's search for the polynomial of an algebraic number whose decimal approximation was known to a certain accuracy, both the accuracy and a constant multiplier related to the algorithm were adjustable. It was never clear whether the constant multiplier was big enough, or the accuracy good enough, to have confidence in the output. If the output stabilized with increasing exactness in approximation and increasing constant multiplier, it was hoped that the actual answer had been reached. This turned out to be true for the few cases LLL was applied to.

Another way to get radical expressions for the singular moduli was to use Bailey and Ferguson's partial sum of squares (PSOS) algorithm (see [1]). This algorithm can take a sufficiently accurate decimal approximation of a root of a polynomial with rational integer coefficients and output the polynomial. Moreover, when the algorithm hits the relevant polynomial, certain estimates of error move down to "machine zero" in use at the time, thus pointing to the most likely polynomial among the several under consideration.

Since the PSOS algorithm was coded up in MACSYMA, the fixed-point precision ability of MACSYMA was used to adjust the machine zero in use. This turned out to be extremely helpful. Often, 30-decimal-digit accuracy was sufficient for PSOS to obtain the polynomial, though sometimes 50 decimal digits were used. Once 16-digit standard double precision was sufficient to obtain the polynomial in question.

Unfortunately, such need for good approximations for singular moduli created some problems. Expansions of degree of accuracy up to 35 were used to try to get a sufficient number of correct digits. Fixed-point precision was set at 105 decimal digits in all calculations. The number of terms in an expansion of accuracy 35 was approximately 2180 terms for functions in H^+ and took so much time that the MACSYMA time counter became negative. The time needed for such a run was estimated at 36 hours. It was necessary to make a table of coefficient values, which reduced computation time for numbers corresponding to points in H^+ to about 1.5 hours. To check accuracy, some numbers were calculated at degree of accuracy 21 or 29 first, to see if the decimal approximations were converging. Sometimes calculations were done at points equivalent under modular transformations. This raised the question of regions of good convergence for Q, R expansions of modular forms, though it gave great confidence to the extent of agreement with estimation at the original point. Also, sometimes trivial extraneous roots such as 0 or 1 showed up with the polynomial for the singular moduli. This may have been due to setting the highest degree possible for the output polynomial of the PSOS algorithm higher than was actually necessary. PSOS runs generally took under 15 minutes, but sometimes there was not enough accuracy to obtain the answer.

Parenthetically, we might expect that the evaluation of singular moduli *together with their conjugates* could suffice to determine exact equations and exact radicals (as in the classical case over $\mathbf{Q}$). Actually, the situation is slightly different here, since in theory the equations are over $\mathbf{Q}(\sqrt{3})$, so that recourse to a PSOS program may be theoretically unavoidable.

8. The Hecke-Weber method

The original mission of Hecke's dissertation [9] was presumably to create an analogue with Hilbert modular functions for the class field theory of (classical) modular functions in one variable. The purpose was not fully achieved but served as the basis of some parallel theories (see [3] and [14]).

We consider the Hecke generalization of Weber's method in a limited context of a real quadratic field

$$k = \mathbf{Q}(\sqrt{M}), \qquad M \in \mathbf{Z}^+, \tag{8.1a}$$

where k is of class number unity. Within the (symmetric) Hilbert modular function field F_k we consider only the modular functions generated by modular forms whose Fourier series have coefficients in k. The *singular moduli* are values of these modular functions whose arguments are totally complex quadratic surds over k. The purpose is to use some of these moduli to generate *ring class fields*.

The ring class fields have an existence independent of Hilbert modular functions. We consider a quadratic form over O_k, the ring of integers of k, written here as (say)

$$\Phi(\xi, \eta) = \xi^2 + A\eta^2, \qquad \xi, \eta \in O_k. \tag{8.1b}$$

Here, $A \in O_k$ is a constant with the property that it is totally positive and $\sqrt{A/A'}$ is a unit of k. (Thus, without any difficulty, any number represented by Φ has a conjugate which is also so represented.) We restrict the representations to quadratic primes. Let $p\ (= \pi\pi')$ be a prime which factors into totally positive factors in k (as shown). Then *the ring class field* $RCF(\Phi)$ is defined by the property that for *some* number π of norm p

$$\{p \text{ splits completely in } RCF(\Phi)\} \equiv \{\pi = \Phi(\xi, \eta), \quad \xi, \eta \in O_k\}. \tag{8.1c}$$

(We of course omit a finite set of primes such as discriminantal divisors.) The restrictions on A may seem cumbersome but they are convenient and in some sense seem necessary (based on unpublished computations). From the general theory [3], $RCF(\Phi)$ is abelian over k_0, the splitting field of Φ over k. This is

$$k_0 = k(\sqrt{-A}) \tag{8.1d}$$

(clearly the same for A as for A'). Indeed, by the symmetry of the form, the ring class field is normal over $\mathbf{Q}$, so that we are dealing with the "self-dual" case (see [2, 12, 14]).

We now define $SMF(\Phi)$, the *singular moduli field* for Φ, as

$$SMF(\Phi) = k_0(U_1(\pm\sqrt{-A}, \pm\sqrt{-A'}), U_2(\pm\sqrt{-A}, \pm\sqrt{-A'}), \ldots), \tag{8.1e}$$

where U_r are the generating functions of F_k and the signs accord with the configuration of half-planes.

From Weber's model (with $\mathbf{Q}$ instead of k) we might expect that $RCF(\Phi) = SMF(\Phi)$, even when Φ is not a form with fundamental discriminant. In Hecke's extension of the theory this is almost the case:

$$SMF(\Phi) \subseteq RCF(\Phi) \tag{8.2a}$$

(which implies that $SMF(\Phi)$ is abelian over k_0, hence normal over $\mathbf{Q}$). In a sense, the two fields are not so far apart:

$$[RCF(\Phi) : SMF(\Phi)] \leq \text{const}, \tag{8.2b}$$

with fixed constant for each k if the number of distinct primes dividing the discriminant of Φ is bounded. There seems to be much numerical evidence for this type of bound (see [3, 5]).

9. Ring class field theory (singular moduli)

We now ask the typical question of ring class field theory. In which field (RCF) does the prime p have to split completely in order that its totally positive prime factor π (or π') be representable as each of the following:

(9.1a) $$A=1:\ \Phi_m[1] = \xi^2 + \eta^2,\quad 2^m|N\xi N\eta,$$

(9.1b) $$A=2:\ \Phi_m[2] = \xi^2 + 2\eta^2,\quad 2^{m-1}|N\eta,$$

(9.1c) $$A=3:\ \Phi_m[3] = \xi^2 + 3\eta^2,\quad 2^m|N\xi N\eta,$$

(9.1d) $$A=6:\ \Phi_m[6] = \xi^2 + 6\eta^2,\quad 2^{m-1}|N\eta.$$

Here, $m > 0$. We search for singular moduli from which to construct the RCF.

To make the problem appear more "conventional", we might rewrite

(9.2a) $$\Phi_m[1]:\ \pi = \xi^2 + \tau^{2m}\eta^2,$$

(9.2b) $$\Phi_m[2]:\ \pi = \xi^2 + 2\tau^{2m-2}\eta^2,$$

(9.2c) $$\Phi_m[3]:\ \pi = \begin{cases} \xi^2 + 3\tau^{2m}\eta^2, \\ \tau^{2m}\xi^2 + 3\eta^2, \end{cases}$$

(9.2d) $$\Phi_m[6]:\ \pi = \xi^2 + 6\tau^{2m-2}\eta^2.$$

The cases $A = 1$ and $A = 2$ have intricate interrelations (since ε is a totally positive nonsquare unit). Thus, in the sense of *improper* equivalence $(GL_2(O_k))$,

(9.3a) $$A=1:\ \Phi_m[1] = \begin{cases} \xi^2 + 2^m\eta^2 & \text{even } m, \\ \xi^2 + 2^m\varepsilon\eta^2 & \text{odd } m, \end{cases}$$

(9.3b) $$A=2:\ \Phi_m[2] = \begin{cases} \xi^2 + 2^m\eta^2 & \text{odd } m, \\ \xi^2 + 2^m\varepsilon\eta^2 & \text{even } m. \end{cases}$$

(Similar relations exist between the cases $A = 3$ and $A = 6$.)

To identify the $RCF(\Phi_m)$, we must initialize with $RCF(\Phi_1)$. We start with the splitting fields of $\Phi_1[A]$. We indicate with each the values of p which split over each field.

(9.4a) $$A=1,\ 3:\ k_0 = \mathbf{Q}(\sqrt{3},\, i) \Leftrightarrow p \equiv 1 \bmod 12,$$

(9.4b) $$A=2,\ 6:\ k_0 = \mathbf{Q}(\sqrt{3},\, \sqrt{-2}) \Leftrightarrow p \equiv 1,\ 11 \bmod 24.$$

Since $\mathbf{Q}(\sqrt{3},\, i)$ has class number unity, the splitting primes (9.4a) have principal factors, so that

(9.4c) $$RCF(\Phi_1[1]) = \mathbf{Q}(\sqrt{3},\, i).$$

Now $\mathbf{Q}(\sqrt{3}, \sqrt{-2})$ has class number two, but the primes which split into principal factors are the subset of (9.4b),

(9.4d) $$p \equiv 1 \bmod 24.$$

Then an elementary manipulation yields the result

(9.4e) $$RCF(\Phi_1[2]) = \mathbf{Q}(\sqrt{3}, \sqrt{2}, i).$$

The cases $A = 3$ and $A = 6$ run along parallel lines, with the new adjunction $3^{1/4}/\sqrt{2}$ (which occurs in a singular moduli field for $A = 3$ but not $A = 6$). This is balanced by the fact that there is an as yet unknown radical required to distinguish the two forms in (9.2c) for $A = 3$.

If we examine the sequences in §6, we see that these are by definition the singular moduli fields:

(9.5a) $$K_n[A] = K_n = k_0(v_n, x_n^*) = k_0(x_n, w_n^*) = k_0(t_n, t_n^*) = k_0(s_{n-1}, s_{n-1}^*).$$

Now the " new" irrationality is determined by (s_{n-1}, s_{n-1}^*), but some of the early values may already be in k_0. We can, however, recognize as a *stable* state

(9.5b) $$|K_{n+1}/K_n| = 4, \qquad n > \text{ const.}$$

An intermediate field $K_n^*[A]$ $(= K_n^*)$ will be of considerable value later on,

(9.5c) $$K_n \subset K_n^* = K_n(s_n s_n^*) \subset K_{n+1} = K_n(s_n, s_n^*),$$

(9.5d) $$[K_{n+1} : K_n^*] = [K_n^* : K_n] = 2, \qquad n > \text{ const.}$$

10. Ring class field theory (verification)

To identify the singular moduli fields is just a matter of comparison of equations of §6 with (9.5a). Generally, the singular moduli field $SMF(\Phi_m)$ is only a subfield of $RCF(\Phi_m)$. Also the necessary adjunction to SMF to make RCF comes from some later $SMF(\Phi_M)$ where $M > m$ (except when $A = 6$).

A supplementary computation of the "BASIC" type is required to identify the $RCF(\Phi_m[A])$. We use the elementary principle that if two fields satisfy $K \supseteq L$, then to show $K = L$ we need only compare degrees. We therefore (by (8.2b)) hunt for the $RCF(\Phi_m[A])$ among the $SMF(\Phi_M[A])$ for $M \geq m$. This is done by representing π by the forms (9.1a–d) and seeing the exact value of m determined by each representation. At the same time we see which values of s_n and s_n^* are "rational" modulo p, i.e., which values have quadratic residues in the radicand as square roots are taken in (5.2f) and (5.4f). When s_{n-1} or s_{n-1}^* fails to be a square root modulo p, then by (9.5a), $K_n[A]$ is not a ring class field, but we test the residuacity of $s_{n-1}s_{n-1}^*$ to verify $K_{n-1}^*[A]$, by (9.5c). The results for $A = 1, 2, 3, 6$ are shown in the Tables I–IV. On the basis of p up to 3000, we get enough cases to identify the $RCF(\Phi_m[A])$.

As a check on the accuracy, the degree of the ring class field is checked against the "density of representations". The ring class field always doubles in degree when we go from Φ_m to Φ_{m+1} (even if the singular moduli do not always create higher fields). The singular moduli fields on the other hand, generally

TABLE I. Results for $A = 1: \Phi_m = \xi^2 + \tau^{2m}\eta^2$

Points on Curves for $\mathbf{A = 1}$ (Sect. 5 and 6)

$$D: \ v_0 = 12, x_0 = 108, \ v_1 = 21, \ x_1 = 1458, \ v_2 = (690 + 297\sqrt{6})/2,$$

$$D \cdot \ s_0 = 3, \ (t_0 = -\frac{3}{5}), \ s_1 = \frac{\sqrt{6}}{2}, \ (t_1 = \frac{3}{5}), \ s_2 = \frac{3}{\sqrt{2(2+\sqrt{6})}}, \ (t_2 = \frac{3+6\sqrt{6}}{13+2\sqrt{6}}),$$

$$D^* : \ x_0^* = 324, w_0^* = 1728, \ x_1^* = 324, \ w_1^* = 287496, \ x_2^* = 71604 + 50787\sqrt{2},$$

$$D^* : \ s_0^* = \frac{1}{3}, (t_0^* = -\frac{7}{9}), \ s_1^* = 2\frac{\sqrt{2}}{3}, (t_1^* = \frac{7}{9}), \ s_2^* = \frac{4(1+\sqrt{2})2^{1/4}}{3(2\sqrt{2}+1)}, (t_2^* = \frac{47+60\sqrt{2}}{81+36\sqrt{2}}),$$

Singular Moduli Fields (Sect. 8 and 9)

$$k_0 = \mathbf{Q}(\sqrt{3}, \imath)$$

$$SMF(\Phi_m) = K_{[(m+1)/2]}$$

$$[K_n : \mathbf{Q}] = 2 \ \ 4^{n-1}, \ \ (n > 1)$$

$$SMF(\Phi_1) = SMF(\Phi_2) = K_1 = k_0$$

$$SMF(\Phi_3) = SMF(\Phi_4) = K_2 = k_0(\sqrt{2})$$

$$SMF(\Phi_5) = SMF(\Phi_6) = K_3 = k_0(2^{1/4}, \sqrt{2+\sqrt{6}})$$

$$[SMF(\Phi_m) : \mathbf{Q}] = 2 \ \ 4^{[(m-1)/2]}$$

Ring Class Fields (Sect. 8 and 9)

$$RCF(\Phi_1) = k_0$$

$$[RCF(\Phi_m) : \mathbf{Q}] = 2[RCF(\Phi_{m-1}) : \mathbf{Q}] = \quad = 2 \ \ 2^m$$

$$m \text{ even } <=> \ RCF(\Phi_m) = SMF(\Phi_{m+1}) = K_{1+m/2}$$

$$m \text{ odd } <=> \ SMF(\Phi_m) \subset RCF(\Phi_m) \subset SMF(\Phi_{m+2})$$

$$m \text{ odd } <=> \ RCF(\Phi_m) = K^*_{(m+1)/2}$$

Representation of Forms (Sect. 10)

$$p = N\pi \ (\equiv 1 \bmod 12), \ \pi = \xi^2 + \eta^2, \ \ 2^m || N\eta, \ (m > 0),$$

m	p	π	ξ	η	p splits
1	13	$5 + 2\sqrt{3}$	1	$1 + \sqrt{3}$	$K_1^* : s_1 s_1^*$
2	73	$11 + 4\sqrt{3}$	$2 + \sqrt{3}$	2	$K_2 : s_1$ and s_1^*
3	97	$17 + 8\sqrt{3}$	1	$2 + 2\sqrt{3}$	$K_2^* : s_2 s_2^*$
4	937	$37 + 12\sqrt{3}$	$3 + 2\sqrt{3}$	4	$K_3 : s_2$ and s_2^*
5	1153	$65 + 32\sqrt{3}$	1	$4 + 4\sqrt{3}$	$K_3^* : s_3 s_3^*$

TABLE II. Results for $A = 2: \Phi_m = \xi^2 + 2\tau^{2m}\eta^2$

Points on Curves for $\mathbf{A = 2}$ (Sect. 5 and 6)

$$D: \ v_0 = 12, \ x_0 = 216, \ v_1 = 39 + 27\sqrt{3}, \ x_1 = 54(265 + 153\sqrt{3}),$$

$$D: \ s_0 = \sqrt{3}, \ (t_0 = 0), \ s_1 = \sqrt{\frac{3}{1+\sqrt{3}}}, \ (t_1 = \frac{3\sqrt{3}}{4+\sqrt{3}}),$$

$$D^* : \ x_0^* = 128, \ w_0^* = 8000, \ x_1^* = 1828 + 1300\sqrt{2}, \ w_1^* = 1000(26125 + 18473\sqrt{2}),$$

$$D^* : \ s_0^* = \sqrt{\frac{1}{2}}, \ (t_0^* = 0), \ s_1^* = \frac{\sqrt{8(1+\sqrt{2})}}{3+\sqrt{2}}, \ (t_1^* = \frac{5(3-\sqrt{2})(1+\sqrt{2})}{(3+\sqrt{2})^2}),$$

Singular Moduli Fields (Sect. 8 and 9)

$$k_0 = \mathbf{Q}(\sqrt{3}, \sqrt{-2})$$

$$SMF(\Phi_m) = K_{[m/2]}$$

$$[K_n : \mathbf{Q}] = 2 \ \ 4^n, \ \ (n > 0)$$

$$SMF(\Phi_1) = K_0 = k_0, \ \ SMF(\Phi_2) = SMF(\Phi_3) = K_1 = k_0(\sqrt{2})$$

$$K_2 = SMF(\Phi_4) = SMF(\Phi_5) = k_0(\sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{3}})$$

$$[SMF(\Phi_m) : \mathbf{Q}] = 2 \ \ 4^{[1+m/2]}, \ \ (m > 1)$$

Ring Class Fields (Sect. 8 and 9)

$$RCF(\Phi_1) = K_1$$

$$[RCF(\Phi_m) : \mathbf{Q}] = 2[RCF(\Phi_{m-1}) : \mathbf{Q}] = \quad = 4 \ \ 2^m$$

$$(1 <) \ m \text{ odd } <=> \ RCF(\Phi_m) = SMF(\Phi_m) = K_{(m+1)/2}$$

$$m \text{ even } <=> \ SMF(\Phi_{m-1}) \subset RCF(\Phi_m) \subset SMF(\Phi_m)$$

$$m \text{ even } <=> \ RCF(\Phi_m) = K^*_{m/2}$$

Representations by Forms (Sect. 10)

$$p = N\pi \ (\equiv 1 \bmod 24), \ \pi = \xi^2 + 2\eta^2, \ \ 2^{m-1} || N\eta,$$

m	p	π	ξ	η	p splits
1	97	$17 + 8\sqrt{3}$	$\sqrt{3}$	$2 + \sqrt{3}$	$K_1 : s_0$ and s_0^*
2	73	$11 + 4\sqrt{3}$	$\sqrt{3}$	$1 + \sqrt{3}$	$K_1^* : s_1 s_1^*$
3	409	$29 + 12\sqrt{3}$	$3 + 2\sqrt{3}$	2	$K_2 : s_1$ and s_1^*
4	457	$35 + 16\sqrt{3}$	$\sqrt{3}$	$2 + 2\sqrt{3}$	$K_2^* : s_2 s_2^*$
5	2377	$53 + 12\sqrt{3}$	$3 + 2\sqrt{3}$	4	$K_3 : s_2$ and s_2^*

TABLE III. Results for $A = 3$: $\Phi_m = \xi^2 + 3\tau^{2m}\eta^2$ or $\tau^{2m}\xi^2 + 3\eta^2$

Points on Curves for A = 3 (Sect. 5 and 6)

$$D:\ v_0 = -24,\ x_0 = 288 + 168\sqrt{3},$$

$$D:\ s_0 = (-1+\sqrt{3})\sqrt{\frac{\sqrt{3}}{2}},\ (t_0 = \sqrt{3}),\ s_1 = \frac{3^{1/4}}{\sqrt{1 + (1-\sqrt{3})3^{1/4}/\sqrt{2}}},$$

$$D^*:\ x_0^* = 18(746 - 425\sqrt{3}),\ w_0^* = 54000,$$

$$D^*:\ s_0^* = (15 + 8\sqrt{3})/33,\ (t_0^* = (-255 + 480\sqrt{3})/1089),\ s_1^* = 2\sqrt{2/3}(5-\sqrt{3})/11,$$

Singular Moduli Fields (Sect. 8 and 9)

$$k_0 = \mathbf{Q}(\sqrt{3}, i)$$
$$SMF(\Phi_m) = K_{[(m+1)/2]}$$
$$[K_n : \mathbf{Q}] = 2\ \ 4^n$$

$$SMF(\Phi_1) = SMF(\Phi_2) = K_1 = k_0(\rho),\ \rho = \sqrt{\sqrt{3}/2}$$
$$SMF(\Phi_3) = SMF(\Phi_4) = K_2 = K_1(\sqrt{2}, \sqrt{1 + (1-\sqrt{3})\rho})$$
$$[SMF(\Phi_m) : \mathbf{Q}] = 4\ \ 4^{[(m-1)/2]},\ (m > 2)$$

Ring Class Fields (Sect. 8 and 9)

$$RCF(\Phi_1) = K_1$$
$$[RCF(\Phi_m) : \mathbf{Q}] = 2[RCF(\Phi_{m-1}) : \mathbf{Q}] = \quad = 4\ \ 2^m$$
$$m \text{ even } <=> RCF(\Phi_m) = SMF(\Phi_{m+1}) = K_{1+m/2}$$
$$(1 <)\ m \text{ odd } <=> SMF(\Phi_m) \subset RCF(\Phi_m) \subset SMF(\Phi_{m+2})$$
$$(1 <)\ m \text{ odd } <=> RCF(\Phi_m) = K^*_{(1+m)/2}$$

Representation of Forms (Sect. 10)

$p = N\pi$ $(\equiv 1 \bmod 12)$, $\pi = \xi^2 + 3\eta^2$, $2^m || N\xi N\eta$, $(m > 0)$,

m	p	π	ξ	η	p splits
1	37	$7+2\sqrt{3}$	$1+\sqrt{3}$	1	$K_1^* : s_1 s_1^*$
1	61	$13+6\sqrt{3}$	1	$1+\sqrt{3}$	$K_1^* : s_1 s_1^*$
2	193	$25+12\sqrt{3}$	2	$2+\sqrt{3}$	$K_2 : s_1$ and s_1^*
2	313	$19+4\sqrt{3}$	$2+\sqrt{3}$	2	$K_2 : s_1$ and s_1^*
3	433	$25+8\sqrt{3}$	$2+2\sqrt{3}$	$\sqrt{3}$	$K_2^* \cdot s_2 s_2^*$
3	673	$49+24\sqrt{3}$	1	$2+2\sqrt{3}$	$K_2^* : s_2 s_2^*$
4	937	$37+12\sqrt{3}$	4	$2+\sqrt{3}$	$K_3 : s_2$ and s_2^*

TABLE IV. Results for $A = 6$: $\Phi_m = \xi^2 + 6\tau^{2m-2}\eta^2$

Points on Curves for A = 6 (Sect. 5 and 6)

$$D:\ v_0 = 44,\ x_0 = 3704 + 1456\sqrt{6},$$

$$D:\ s_0 = 2\sqrt{2} - \sqrt{3},\ (t_0 = \sqrt{6}/3),\ s_1 = (\sqrt{6}-1)/\sqrt{1 + 2\sqrt{2} - \sqrt{3}},$$

$$D^*:\ x_0^* = 1152,\ w_0^* = 1728(1399 + 988\sqrt{2}),$$

$$D^*:\ s_0^* = \frac{1+\sqrt{2}}{\sqrt{6}},\ (t_0^* = \frac{2\sqrt{2}}{3}),\ s_1^* = \frac{2\sqrt{2(1+\sqrt{2})(1+\sqrt{2}+\sqrt{6})}}{\sqrt{3}(\sqrt{2}+\sqrt{3}+\sqrt{6})},$$

Singular Moduli Fields (Sect. 8 and 9)

$$k_0 = \mathbf{Q}(\sqrt{3}, \sqrt{-2})$$
$$SMF(\Phi_m) = K_{[m/2]}$$
$$[K_n : \mathbf{Q}] = 2\ \ 4^n,\ (n > 0)$$

$$SMF(\Phi_1) = k_0(\sqrt{2}) = K_0 = SMF(\Phi_2) = SMF(\Phi_3) = K_1$$
$$K_2 = SMF(\Phi_4) = SMF(\Phi_5) = k_0\left(\frac{\sqrt{6}-1}{\sqrt{1+2\sqrt{2}-\sqrt{3}}}, \frac{2\sqrt{2(1+\sqrt{2})(1+\sqrt{2}+\sqrt{6})}}{\sqrt{3}(\sqrt{2}+\sqrt{3}+\sqrt{6})}\right)$$
$$[SMF(\Phi_m) : \mathbf{Q}] = 2\ \ 4^{[m/2]}$$

Ring Class Fields (Sect. 8 and 9)

$$RCF(\Phi_1) = K_0(\lambda),\ \lambda = 3^{1/4}$$
$$[RCF(\Phi_m) : \mathbf{Q}] = 2[RCF(\Phi_{m-1}) : \mathbf{Q}] = \quad = 8\ \ 2^m$$
$$m \text{ odd } <=> RCF(\Phi_m) = SMF(\Phi_{m+1})(\lambda) = K_{(m+1)/2}(\lambda)$$
$$m \text{ even } <=> SMF(\Phi_m)(\lambda) \subset RCF(\Phi_m) \subset SMF(\Phi_{m+2})(\lambda)$$
$$m \text{ even } <=> RCF(\Phi_m) = K^*_{m/2}(\lambda).$$

Representations by Forms (Sect. 10)

$p = N\pi$ $(\equiv 1 \bmod 24)$, $\pi = \xi^2 + 6\eta^2$, $2^{m-1} || N\eta$,

m	p	π	ξ	η	p splits λ and
1	313	$19+4\sqrt{3}$	$1+2\sqrt{3}$	1	$K_1 : s_0$ and s_0*
2	193	$25+12\sqrt{3}$	$\sqrt{3}$	$1+\sqrt{3}$	$K_1^* : s_1 s_1^*$
3	1321	$37+4\sqrt{3}$	$1+2\sqrt{3}$	2	$K_2 : s_1$ and s_1^*

increase in degree by a factor of four as $m \to m+2$, but they can be identified with ring class fields by degree.

11. Concluding remarks

The fact that we are on a curve within the surface (2.5a–d) makes a one-dimensional theory possible, analogously with the earlier work on $\mathbf{Q}(\sqrt{2})$, see [5]. We omit the details for now, but we note that the Klein and Hecke modular functions produce the results

$$K_n[1] = \mathbf{Q}(i, \sqrt{-3}, j(2^n i), j(2^n\sqrt{-3})), \tag{11.1a}$$

$$K_n[2] = \mathbf{Q}(\sqrt{-2}, \sqrt{3}, j(2^n\sqrt{-2}), j(2^n\sqrt{-6})). \tag{11.1b}$$

The role of curves on modular manifolds (defined by the modular function fields) has been a fruitful topic of exploration (see [7, 10]). Certainly, the numerical properties of the curves used here should indicate the importance of a more algebraic approach to justify the plethora of useful numerical data available even now, particularly with MACSYMA.

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